

Weak Value in Wave Function of Detector

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“Weak measurement” and its outcome, “weak value”, are topical subjects in the recent development of quantum mechanics.¹⁻⁴⁾ This weak measurement is not only a theoretical concept but also applied to many experiments.^{5,6)} In particular, through the weak measurement, we have an amplification of the outcome which is much larger than the eigenvalue spectrum of the observable. Therefore, the properties of weak measurements and their applications are interesting issues in recent physics.

In the weak measurement, the total system consists of the “system” and the “detector”. The system is a quantum system and we want to measure an observable of this system. The detector is also a quantum system that interacts with the system for measuring an observable of the system. In some cases, the wave function of the detector after the weak measurement has a complicated form, even when we know the explicit analytical expression of the wave function.⁴⁾ In such cases, it is tedious to derive the outcome, i.e., “weak value” from complicated wave functions. Therefore, a simple formula to derive the weak value in the explicit wave function is useful. In this short note, we propose a simple theoretical formula for deriving the weak value from the explicit wave function of the detector after the weak measurement.

The process of the weak measurement consists of a sequence of four measurements.¹⁾ The first three processes of these four measurements are called “preselection”, “weak measurement”, and “postselection”. The final one is the measurement of the pointer variable of the detector by any type of applicable measurement in quantum mechanics.

First, we prepare the initial state $|\Psi\rangle$ of the system through the projection measurement at $t = t_1$, which is called “preselection”. We also prepare the initial state of the detector $|\Psi_{in}^D\rangle$, and the total system is described by the state $|\Psi\rangle \otimes |\Psi_{in}^D\rangle$. In this note, we assume that the initial wave function $\Psi_{in}^D := \langle Q | \Psi_{in}^D \rangle$ of the detector is a Gaussian distribution

$$\Psi_{in}^D = \left(\frac{\beta}{\pi}\right)^{1/4} \exp\left(-\frac{\beta Q^2}{2}\right), \quad (1)$$

where Q is the pointer variable of the detector and $\beta^{-1/2}$ is the width of this Gaussian distribution.

Second, after the preselection at $t = t_1$, the switch of the interaction between the system and the detector is turned on and the detector measures the observable C of

the system through this interaction. Typically, this interaction is described by the von Neumann measurement⁷⁾ with the interaction Hamiltonian

$$H = g(t)PC, \quad (2)$$

where P is the conjugate momentum to the pointer variable Q and the function $g(t)$ is the switch of the interaction with the support $t \in [t_1, t_2]$, $t_1 < t_2$; $g(t) = 0$ outside of this support. In some papers, the “weak” of the weak measurement means $g(t)$ is sufficiently small and then the disturbance to the system from the measurement by the detector is negligible. However, this coupling constant is completely determined when we specify the concrete system and the detector, and we cannot control it after this specification. Therefore, in this note, we choose $\int_{t_1}^{t_2} g(t) dt = 1$. Instead, we choose β so that $\beta^{-1/2}$ is sufficiently large, as the original proposal of the weak measurement.¹⁾ β is controllable through the preparation of the initial state of the detector, and this choice yields negligible disturbance to the system through the measurement, and then, so called “weak measurement” is accomplished.

Next, we perform “postselection”, which is the projection measurement of the system to the state $|\Phi\rangle$ at $t = t_2$. It is important to note that we choose the postselected state $|\Phi\rangle$ such that $|\Phi\rangle$ is not orthogonal to the preselected state $|\Psi\rangle$ of the system ($\langle \Phi | \Psi \rangle \neq 0$) but nearly orthogonal. Then, the state $|\Psi_{out}^D\rangle$ of the detector after the postselection is given by

$$|\Psi_{out}^D\rangle = \langle \Phi | \exp\left(-i \int_{t_1}^{t_2} dt' H(t')\right) |\Psi\rangle \otimes |\Psi_{in}^D\rangle. \quad (3)$$

As shown by Aharonov and Vaidmann,³⁾ the wave function $\Psi_{out}^D := \langle Q | \Psi_{out}^D \rangle$ of the detector after postselection is given by

$$\begin{aligned} \Psi_{out}^D &\propto \exp\left\{-\frac{\beta}{2}(Q - C_w)^2\right\} \\ &+ e^{-\frac{\beta Q^2}{2}} \sum_{n=2}^{\infty} \frac{(C^n)_w - (C_w)^n}{n! \sqrt{\pi}} \left(-\sqrt{2\beta}i\right)^n \\ &\times \int_{-\infty}^{\infty} dx \left(x + \frac{i\beta^{1/2}Q}{\sqrt{2}}\right)^n e^{-x^2}, \quad (4) \end{aligned}$$

where C_w is called a “weak value” and is defined by

$$C_w := \frac{\langle \Phi | C | \Psi \rangle}{\langle \Phi | \Psi \rangle} \quad (5)$$

and $(C^n)_w := \langle \Phi | C^n | \Psi \rangle / \langle \Phi | \Psi \rangle$.

If the interaction between the system and the detector is really “weak”, this sequence of the measurements yields the shift of the pointer variable Q of the detector by the weak value C_w . This can be easily seen from eq. (4). In our “weak” measurement, $\beta^{-1/2}$ is sufficiently large, i.e., the initial wave function of the detector has a broad profile. Then the second term in the right-hand side of eq. (4) is negligible. Because of the weakness of the measurement, a single weak measurement of the detector may not yield this weak value, i.e., the measurement is imprecise. However, by performing the weak measure-

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ment on an ensemble of N identical systems, this imprecision is improved by the factor \sqrt{N} . This is the success of the weak measurement. Here, we must note that, according to the value of $\beta^{-1/2}$ and the pre- and postselections, the weak measurement may fail to succeed and we may not obtain the weak value as the shift in Ψ_{out}^D . In this case, the second term on the right-hand side of eq. (4) is not negligible.

Now, we propose a simple formula for calculating the weak value from the wave function of the detector after the postselection in weak measurement. To extract the weak value from the wave function (4), the overall normalization of the wave function is irrelevant. To extract the information that is independent of this overall normalization, we consider the partial derivative of the logarithm of eq. (4) with respect to Q . Furthermore, from the order counting of β , we can expect that the derivative of the logarithm of eq. (4) with respect to β is related to the weak value C_w . Actually, we can easily show the formula

$$\frac{\partial}{\partial \beta} \left[\frac{\partial}{\partial Q} \ln(\Psi_{out}^D(Q)) \right] \Big|_{Q=0} \Big|_{\beta=0} = C_w. \quad (6)$$

Through this formula, if the initial wave function of the detector is Gaussian, we can extract the weak value from the wave function of the detector after the postselection, even if the wave function has a complicated form. In eq. (6), $\beta = 0$ corresponds to the weak limit of the measurement, i.e., the second term in eq. (4) does not affect to the calculation using eq. (6).

Next, as an example, we explicitly derive the weak value in a concrete experiment using the formula (6). The experiment that we consider here is an optical analog of the Stern-Gerlach spin- $\frac{1}{2}$ experiment proposed by Duck et al.,²⁾ and carried out by Ritchie et al.⁵⁾ In this analog, the beam of spin- $\frac{1}{2}$ particles is replaced by a Gaussian-mode laser beam and the preselection and postselection Stern-Gerlach magnets are replaced by optical polarizers. The weak measurement is performed using a birefringent-crystalline quartz plate that spatially separates the two orthogonal polarizations of the laser radiation by a distance much less than the Gaussian beam waist of the laser beam.

Now, we consider the laser to be propagating along the z direction and to be linearly polarized at an angle α with respect to the x axis as the preselection. The electric-field vector of this radiation is described by

$$\mathbf{E}_i = E_0 \exp\left(-\frac{\beta(x^2 + y^2)}{2}\right) (\cos \alpha \hat{\mathbf{x}} + \sin \alpha \hat{\mathbf{y}}), \quad (7)$$

where $\beta^{-1/2}$ is the beam waist. In this experiment, the system in the weak measurement is the polarization of the laser beam and the detector in the weak measurement is the transverse distribution of the laser beam. Therefore, eq. (7) shows that the initial state of the detector has the Gaussian distribution. The laser beam is incident on a plane-parallel uniaxial birefringent plate whose optic axis is aligned with the x axis. The plane of the plate includes the x axis and is rotated from the y

axis by angle θ . The birefringent plate performs a weak measurement by spatially separating the two orthogonal linear-polarization components of the field, corresponding to the ordinary and extraordinary rays,⁸⁾ by a distance a which is determined by θ and is small compared with $\beta^{-1/2}$. After this weak measurement, the postselection is performed by a polarizer aligned at angle α' with respect to the x -axis. Then, the wave function of the detector after this postselection is given by

$$\Psi^D(y) \propto \cos \alpha \cos \alpha' e^{-\beta(y+a)^2/2} + \sin \alpha \sin \alpha' e^{-\beta y^2/2}. \quad (8)$$

From this wave function (8) of the detector, we can derive the weak value through formula (6):

$$C_w = -\frac{a}{1 + \tan \alpha \tan \alpha'}. \quad (9)$$

If we choose $\alpha = \pi/4$ and $\alpha' = \alpha + \pi/2 + \epsilon$ with $\epsilon \ll 1$, we have $C_w = -\frac{1}{2}a(1 + \tan \epsilon) \cot \epsilon$, which coincides with the estimated weak value reported by Ritchie et al.⁵⁾

In calculation (9), the interference effect between two Gaussian profiles in eq. (8) is automatically taken into account through the derivative of the logarithm of the wave function. The coexistence of the two terms on the right-hand side of eq. (8) is essential to the nontrivial weak value (9). Thus, through the calculation of the weak value by formula (6), we can easily see that the weak value is realized through the interference of these two Gaussian profiles in this experiment.

In summary, we proposed a theoretical formula for deriving the weak value from the detector wave function after the postselection. We must emphasize that our formula (6) automatically includes the interference effect, which is necessary for the success of the weak measurement, originating from the derivative of the logarithm of the wave function in eq. (6). Therefore, we may say that our formula (6) is a natural form of the weak value and this formula will be useful when we derive the weak value from complicated analytic wave functions of the detector.

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